The almost Gorenstein Rees algebras of socle ideals

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§1 Introduction

History of almost Gorenstein rings

- [Barucci–Fröberg, 1997]
  ⋯ one–dimensional analytically unramified local rings
- [Goto–Matsuoka–Phuong, 2013]
  ⋯ one–dimensional Cohen–Macaulay local rings
- [Goto–Takahashi–T, 2015]
  ⋯ higer dimensional Cohen–Macaulay local/graded rings

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  – the ideal generated by a subsystem of parameters, and
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The main result of this talk is stated as follows.

Theorem 1.2

Let $(R, \mathfrak{m})$ be a RLR with $d = \dim R \geq 3$ and $|R/\mathfrak{m}| = \infty$. Let Q be a parameter ideal of R s.t. $Q \neq \mathfrak{m}$. Set $I = Q : \mathfrak{m}$. Then TFAE.

(1) *$\mathcal{R}(I)$ is an almost Gorenstein graded ring.*

(2) *Either $I = \mathfrak{m}$, or $d = 3$ and $I = (x) + \mathfrak{m}^2$ for $\exists\, x \in \mathfrak{m} \setminus \mathfrak{m}^2$.*

§2 Almost Gorenstein rings

Setting 2.1 (local rings)

- $(R, \mathfrak{m})$ a Cohen–Macaulay local ring with $d = \dim R$
- $\exists$ the canonical module K_R
- $|R/\mathfrak{m}| = \infty$

Definition 2.2

We say that R is *an almost Gorenstein local ring*, if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R–modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$.

Consider an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R–modules. If $C \neq (0)$, then C is Cohen–Macaulay and $\dim_R C = d - 1$.

Set $\overline{R} = R/[(0) :_R C]$.

Then $\exists$ $f_1, f_2, \ldots, f_{d-1} \in \mathfrak{m}$ s.t. $(f_1, f_2, \ldots, f_{d-1})\overline{R}$ forms a minimal reduction of $\overline{\mathfrak{m}} = \mathfrak{m}\overline{R}$.

Therefore

$$\mathrm{e}^0_{\mathfrak{m}}(C) = \mathrm{e}^0_{\overline{\mathfrak{m}}}(C) = \ell_R(C/(f_1, f_2, \ldots, f_{d-1})C) \geq \ell_R(C/\mathfrak{m}C) = \mu_R(C).$$

Thus

$$\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C) \Longleftrightarrow \mathfrak{m}C = (f_1, f_2, \ldots, f_{d-1})C.$$

Hence C is a *maximally generated maximal Cohen–Macaulay $\overline{R}$–module* in the sense of B. Ulrich (cf. [2]), which is called *an Ulrich R–module*.

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Example 2.3

(1) $k[[t^3, t^4, t^5]]$

(2) $k[[t^a, t^{a+1}, \ldots, t^{2a-3}, t^{2a-1}]]$ $(a \geq 4)$

(3) $k[[t^3, t^7, t^8]]$ (this is NOT an almost Gorenstein local ring)

(4) $k[[X, Y, Z]]/(X, Y) \cap (Y, Z) \cap (Z, X)$

(5) Non–Gorenstein almost Gorenstein local ring is G–regular

(6) 1–dimensional finite CM–representation type

(7) 2–dimensional rational singularity

Example 2.4

Let $U = k[[X_1, X_2, \ldots, X_n, Y_1, Y_2, , \ldots, Y_n]]$ $(n \geq 2)$ be the formal power series ring over an infinite field k and put

$$R = U/\mathrm{I}_2(\mathbb{M})$$

where $\mathrm{I}_2(\mathbb{M})$ denotes the ideal of U generated by 2×2 minors of the matrix

$$\mathbb{M} = \begin{pmatrix} X_1 & X_2 & \cdots & X_n \\ Y_1 & Y_2 & \cdots & Y_n \end{pmatrix}.$$

Then R is almost Gorenstein with $\dim R = n + 1$ and $\mathrm{r}(R) = n - 1$.

Here $\mathrm{r}(R)$ stands for the Cohen–Macaulay type of R.

Lemma 2.5

Let R be an almost Gorenstein local ring and choose an exact sequence

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

of R–modules s.t. $\mu_R(C) = \mathrm{e}^0_{\mathfrak{m}}(C)$. If $\varphi(1) \in \mathfrak{m}\,\mathrm{K}_R$, then R is a RLR.

Therefore

$$\mu_R(C) = \mathrm{r}(R) - 1$$

provided R is not a RLR.

Corollary 2.6

Let R be an almost Gorenstein local ring but not Gorenstein. Choose an exact sequence

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

of R–modules s.t. C is an Ulrich R–module.

Then

$$0 \to \mathfrak{m}\varphi(1) \to \mathfrak{m}\,\mathrm{K}_R \to \mathfrak{m}C \to 0$$

is an exact sequence of R–modules.

Hence we get

$$\mu_R(\mathfrak{m}\,\mathrm{K}_R) \leq \mu_R(\mathfrak{m}) + \mu_R(\mathfrak{m}C).$$

Setting 2.7 (graded rings)

- $R = \bigoplus_{n \geq 0} R_n$ a Cohen–Macaulay graded ring with $d = \dim R$
- $(R_0, \mathfrak{m})$ a local ring
- $\exists$ the graded canonical module K_R
- $\mathfrak{M} = \mathfrak{m}R + R_+$
- $a = \mathrm{a}(R) := -\min\{n \in \mathbb{Z} \mid [\mathrm{K}_R]_n \neq (0)\}$
- $|R_0/\mathfrak{m}| = \infty$

Definition 2.8

We say that R is *an almost Gorenstein graded ring*, if $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R(-a) \to C \to 0$$

of graded R–modules such that $\mu_R(C) = \mathrm{e}^0_{\mathfrak{M}}(C)$.

Notice that

- R is an almost Gorenstein graded ring
  $\Longrightarrow R_{\mathfrak{M}}$ is an almost Gorenstein local ring.

Example 2.9

Let $R = k[X_1, X_2, \ldots, X_d]$ $(d \geq 1)$ be a polynomial ring over an infinite field k. Let $n \geq 1$ be an integer.

- $R^{(n)} = k[R_n]$ is an almost Gorenstein graded ring, if $d \leq 2$.
- Suppose that $d \geq 3$. Then $R^{(n)}$ is an almost Gorenstein graded ring if and only if either $n \mid d$, or $d = 3$ and $n = 2$.

Theorem 2.10 ([Goto–Iai, 2000])

Let R be a Gorenstein local ring, $I \subsetneq R$ an ideal of R. If $\mathrm{gr}_I(R)$ is an almost Gorenstein graded ring, then $\mathrm{gr}_I(R)$ is Gorenstein.

Theorem 2.11 ([Goto–Takahashi–T, 2015])

Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 3$ and Q a parameter ideal of R. Then TFAE.

(1) $\mathcal{R}(Q)$ *is an almost Gorenstein graded ring.*

(2) $Q = \mathfrak{m}$.

§3 Main results

In this section

- $(R, \mathfrak{m})$ a Gorenstein local ring with $d = \dim R \geq 3$
- $|R/\mathfrak{m}| = \infty$
- I an $\mathfrak{m}$–primary ideal in R which contains a parameter ideal Q as a reduction (i.e., $\exists\ r \geq 0$ s.t. $I^{r+1} = QI^r$)
- $J := Q : I$
- $\mathcal{R} = \mathcal{R}(I) := R[It] \subseteq R[t]$ the Rees algebra of I
- $\mathfrak{M} := \mathfrak{m}\mathcal{R} + \mathcal{R}_+$ the unique graded maximal ideal of $\mathcal{R}$

Fact 3.1

Suppose that $I^2 = QI$. Then

- [Goto–Shimoda, 1979] $\mathcal{R}$ *is a Cohen–Macaulay ring.*
- [Ulrich, 1996] *One has*

$$\mathrm{K}_{\mathcal{R}}(1) = \sum_{i=0}^{d-3} \mathcal{R}{\cdot}t^i + \mathcal{R}{\cdot}Jt^{d-2}.$$

Note: $\mathrm{a}(\mathcal{R}) = -1$ and $\dim \mathcal{R} = d + 1$.

Let $\mathrm{r}(\mathcal{R})$ denote the Cohen–Macaulay type of $\mathcal{R}$.

Corollary 3.2

Suppose that $I^2 = QI$. *Then*

$$\mathrm{r}(\mathcal{R}) = d - 2 + \mu_R(J/I).$$

In particular, $\mathcal{R}$ *is a Gorenstein ring* $\Longleftrightarrow$ $I = J$ *and* $d = 3$.

Theorem 3.3

Suppose that $J = \mathfrak{m}$ and $I \subseteq \mathfrak{m}^2$. Then $\mathcal{R}_{\mathfrak{M}}$ is NOT an almost Gorenstein local ring.

Note that

- $Q \subseteq \mathfrak{m}^2 \Rightarrow I^2 = QI$
- R is Gorenstein and $J = \mathfrak{m} \Rightarrow I = Q : \mathfrak{m}$

Corollary 3.4

Let Q be a parameter ideal of R s.t. $Q \subseteq \mathfrak{m}^2$. Set $I = Q : \mathfrak{m}$. Then $\mathcal{R}_{\mathfrak{M}}$ is NOT an almost Gorenstein local ring.

Hence $\mathcal{R}(I)$ is NOT an almost Gorenstein graded ring.

Theorem 3.3

Suppose that $J = \mathfrak{m}$ and $I \subseteq \mathfrak{m}^2$. Then $\mathcal{R}_{\mathfrak{M}}$ is NOT an almost Gorenstein local ring.

Note that

- $Q \subseteq \mathfrak{m}^2 \Rightarrow I^2 = QI$
- R is Gorenstein and $J = \mathfrak{m} \Rightarrow I = Q : \mathfrak{m}$

Corollary 3.4

Let Q be a parameter ideal of R s.t. $Q \subseteq \mathfrak{m}^2$. Set $I = Q : \mathfrak{m}$. Then $\mathcal{R}_{\mathfrak{M}}$ is NOT an almost Gorenstein local ring.

Hence $\mathcal{R}(I)$ is *NOT* an almost Gorenstein graded ring.

Proof of Theorem 3.3.

Set $A = \mathcal{R}_{\mathfrak{M}}$ and suppose that A is an almost Gorenstein local ring. Choose an exact sequence

$$0 \to A \xrightarrow{\varphi} \mathrm{K}_A \to C \to 0$$

of A-modules with $C \neq (0)$ and C is an Ulrich A–module. Let $\mathfrak{n}$ denote the maximal ideal of A and take $f_1, f_2, \ldots, f_d \in \mathfrak{n}$ s.t.

$$\mathfrak{n}C = (f_1, f_2, \ldots, f_d)C.$$

Since $\varphi(1) \notin \mathfrak{n}\,\mathrm{K}_A$, we get

$$\mu_A(\mathfrak{n}C) \leq d{\cdot}(r-1)$$

where $r = \mathrm{r}(A) = (d-2) + \mu_R(J/I)$.

We also have

$$0 \to \mathfrak{n}\varphi(1) \to \mathfrak{n}\,\mathrm{K}_A \to \mathfrak{n}C \to 0$$

because $\varphi(1) \notin \mathfrak{n}\,\mathrm{K}_A$.

Hence

$$\begin{aligned} \mu_{\mathcal{R}}(\mathfrak{M}\,\mathrm{K}_{\mathcal{R}}) = \mu_A(\mathfrak{n}\,\mathrm{K}_A) &\leq \mu_A(\mathfrak{n}C) + \mu_A(\mathfrak{n}) \\ &\leq d\cdot[(d-2) + \mu_R(J/I) - 1] + [\mu_R(\mathfrak{m}) + \mu_R(I)]\,. \end{aligned}$$

On the other hand, we have

$$\mu_{\mathcal{R}}(\mathfrak{M}\,\mathrm{K}_{\mathcal{R}}) = (d-2)\cdot\mu_R(\mathfrak{m}) + \mu_R(I + \mathfrak{m}J) + \mu_R(IJ/I^2).$$

Therefore

$$\left[\mu_R(I + \mathfrak{m}J) + \mu_R(IJ/I^2)\right] - \left[\mu_R(I) + d\cdot\mu_R(J/I)\right] \leq (d-3)\cdot[d - \mu_R(\mathfrak{m})] \leq 0.$$

Thus

$$\mu_R(I+\mathfrak{m}J)+\mu_R(IJ/I^2)\le \mu_R(I)+d{\cdot}\mu_R(J/I).$$

Since $J=\mathfrak{m}$, $I\subseteq \mathfrak{m}^2$ and $\mathfrak{m}I=\mathfrak{m}Q$, we get

$$\mu_R(\mathfrak{m}^2)+\mu_R(\mathfrak{m}Q)\le \mu_R(I)+d{\cdot}\mu_R(\mathfrak{m})$$

whence

$$\mu_R(\mathfrak{m}^2)\le \mu_R(I).$$

Hence

$$\binom{d+1}{2}\le \mu_R(\mathfrak{m}^2)\le \mu_R(I)=d+1$$

which is impossible, because $d\ge 3$.

□

Let me recall the main result of this talk.

Theorem 1.2

Suppose that R is a RLR. Let Q be a parameter ideal of R s.t. $Q \neq \mathfrak{m}$. Set $I = Q : \mathfrak{m}$. Then TFAE.

(1) *$\mathcal{R}(I)$ is an almost Gorenstein graded ring.*

(2) *Either $I = \mathfrak{m}$, or $d = 3$ and $I = (x) + \mathfrak{m}^2$ for $\exists\, x \in \mathfrak{m} \setminus \mathfrak{m}^2$.*

Let us begin with the case where $Q \not\subseteq \mathfrak{m}^2$.

Setting 3.5

Let $(R, \mathfrak{m})$ be a RLR, $d = \dim R \geq 3$, $|R/\mathfrak{m}| = \infty$, $\mathfrak{m} = (x_1, x_2, \ldots, x_d)$. Let Q be a parameter ideal of R and let $1 \leq i \leq d-2$ be an integer. Suppose that

$$(x_j \mid 1 \leq j \leq i) \subseteq Q \subseteq (x_j \mid 1 \leq j \leq i) + \mathfrak{m}^2.$$

Set

$$\mathfrak{a} = (x_j \mid 1 \leq j \leq i), \ \mathfrak{b} = (x_j \mid i+1 \leq j \leq d)$$

and $I = Q : \mathfrak{m}$.

Hence

$$Q = \mathfrak{a} + (a_j \mid i+1 \leq j \leq d)$$

with $a_j \in \mathfrak{b}^2$.

Therefore we have the presentation

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ a_{i+1} \\ \vdots \\ a_d \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ & 0 & & \alpha_{jk} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ x_{i+1} \\ \vdots \\ x_d \end{pmatrix}$$

with $\alpha_{jk} \in \mathfrak{b}$ for $i+1 \leq \forall j, k \leq d$.

Let $\Delta = \det(\alpha_{jk})$. Then $\Delta \in \mathfrak{b}^2$ and $Q : \Delta = \mathfrak{m}$, so that

$$I = Q + (\Delta).$$

Proposition 3.6

The following assertions hold true.

(1) $I^2 = QI$.

(2) $Q : I = \mathfrak{m}$.

(3) $I \subseteq \mathfrak{a} + \mathfrak{b}^2$.

(4) $\mu_R(\mathfrak{m}/I) = d - i$ *and hence* $\mathrm{r}(\mathcal{R}) = 2d - (i+2)$.

The following is the heart of the proof of Theorem 1.2.

Proposition 3.7

Suppose that $\mathcal{R}(I)$ is an almost Gorenstein graded ring. Then $d = 3$ and $I = (x_1) + \mathfrak{m}^2$.

Proof of Proposition 3.7.

Since $\mathrm{r}(\mathcal{R}) = 2d - (i+2) \geq 3$, $\mathcal{R}$ is not Gorenstein. Take an exact sequence

$$0 \to \mathcal{R} \xrightarrow{\varphi} \mathrm{K}_{\mathcal{R}}(1) \to C \to 0$$

of graded $\mathcal{R}$–modules s.t. $C \neq (0)$ and C is an Ulrich $\mathcal{R}$–module.

Since $[\mathrm{K}_{\mathcal{R}}]_1 = R$ and $\varphi(1) \notin \mathfrak{M}{\cdot}[\mathrm{K}_{\mathcal{R}}(1)]$, $\varphi(1)$ is a unit of R.

Therefore

$$C \cong \left[\sum_{i=1}^{d-3} \mathcal{R}{\cdot}t^i + \mathcal{R}{\cdot}\mathfrak{m}t^{d-2}\right] / \mathcal{R}_+.$$

Fact.

$$\mu_{\mathcal{R}}(\mathfrak{M}C) = \begin{cases} \mu_R(\mathfrak{m}^2/I \cap \mathfrak{m}^2) + \mu_R(\mathfrak{m}Q/I^2) & (d=3), \\ (d-i) + d(d-4) + \mu_R(I+\mathfrak{m}^2) + \mu_R(\mathfrak{m}Q/I^2) & (d \geq 4). \end{cases}$$

On the other hand, we have

$$\mu_{\mathcal{R}_{\mathfrak{M}}}(C_{\mathfrak{M}}) = \mathrm{r}(\mathcal{R}) - 1 = 2d - (i+3).$$

Consequently

$$\mu_{\mathcal{R}}(\mathfrak{M}C) = \mu_{\mathcal{R}_{\mathfrak{M}}}(\mathfrak{M}C_{\mathfrak{M}}) \leq d \cdot (2d - (i+3)),$$

because $C_{\mathfrak{M}}$ is an Ulrich $\mathcal{R}_{\mathfrak{M}}$–module of dimension d.

Suppose that $d \geq 4$. Then we have

$$(d-i) + d(d-4) + \mu_R(I + \mathfrak{m}^2) + d(d-i) \leq d(2d-(i+3)),$$

and hence

$$\mu_R(I + \mathfrak{m}^2) \leq i$$

which is impossible, because $\mu_R(I + \mathfrak{m}^2) = i + \binom{d-i+1}{2} > i$.

Therefore $d = 3$ and $i = 1$, whence

$$\mu_R(\mathfrak{m}^2/[I \cap \mathfrak{m}^2]) + \mu_R(\mathfrak{m}Q/I^2) \leq 6,$$

so that we have

$$\mathfrak{m}^2 = I \cap \mathfrak{m}^2 \subseteq I$$

because $\mu_R(\mathfrak{m}Q/I^2) = 6$. Hence $I = (x_1) + \mathfrak{m}^2$. □

We need one more result to prove Theorem 1.2.

Proposition 3.8

Let $(R, \mathfrak{m})$ *be a Gorenstein local ring with* $\dim R = 3$ *and* $|R/\mathfrak{m}| = \infty$*.*
Let Q *be a parameter ideal of* R *and set* $I = Q : \mathfrak{m}$*.*
If $I^2 = QI$ *and* $\mathfrak{m}^2 \subseteq I$*, then* $\mathcal{R}(I)$ *is an almost Gorenstein graded ring.*

Theorem 1.2

Suppose that R is a RLR. Let Q be a parameter ideal of R s.t. $Q \neq \mathfrak{m}$. Set $I = Q : \mathfrak{m}$. Then TFAE.

(1) *$\mathcal{R}(I)$ is an almost Gorenstein graded ring.*

(2) *Either $I = \mathfrak{m}$, or $d = 3$ and $I = (x) + \mathfrak{m}^2$ for $\exists\, x \in \mathfrak{m} \setminus \mathfrak{m}^2$.*

Proof of Theorem 1.2.

$(2) \Rightarrow (1)$ If $I = \mathfrak{m}$, then $\mathcal{R}$ is almost Gorenstein by [Goto–Takahashi–T]. Suppose that $d = 3$ and $I = (x) + \mathfrak{m}^2$ for $\exists\, x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Since $\mathfrak{m}^2 \subseteq I$ and $I^2 = QI$, $\mathcal{R}$ is an almost Goreinstein graded ring.

$(1) \Rightarrow (2)$ We have $Q \not\subseteq \mathfrak{m}^2$. If $Q = \overline{Q}$, then

$$Q = (x_1, x_2, \ldots, x_{d-1}, x_d^q)$$

for $\exists$ regular system $\{x_i\}_{1 \le i \le d}$ of parameters of R and for $\exists\, q > 1$. Therefore

$$I = Q : \mathfrak{m} = Q : x_d = (x_1, x_2, \ldots, x_{d-1}, x_d^{q-1})$$

which implies $q = 2$ and $I = \mathfrak{m}$. Suppose that $Q \neq \overline{Q}$. Let $\{x_j\}_{1 \le j \le d}$ be a regular sop of R and choose the integer $1 \le i \le d-2$ s.t.

$$(x_j \mid 1 \le j \le i) \subseteq Q \subseteq (x_j \mid 1 \le j \le i) + \mathfrak{m}^2.$$

Then $d = 3$ and $I = (x_1) + \mathfrak{m}^2$.

Let us note one example.

Example 3.9

Let $R = k[[x, y, z]]$ be the formal power series ring over an infinite field k. We set $\mathfrak{m} = (x, y, z)$, $Q = (x, y^2, z^n)$ with $n \geq 2$, and $I = Q : \mathfrak{m}$.

Then $I = (x, y^2, yz^{n-1}, z^n)$ and $I^2 = QI$.

(1) If $n = 2$, then $I = (x) + \mathfrak{m}^2$, so that $\mathcal{R}(I)$ is an almost Gorenstein graded ring.

(2) Suppose that $n \geq 3$. Then $I \neq \mathfrak{m}$, $Q \neq \mathfrak{m}$, and

$$I \neq (f) + \mathfrak{m}^2$$

for any $f \in \mathfrak{m} \setminus \mathfrak{m}^2$.

Hence $\mathcal{R}(I)$ is ***NOT*** an almost Gorenstein graded ring.

The end of this talk, let us consider the case where $\dim R = 2$.

For each ideal I of R, we set

$$\mathrm{o}(I) = \sup\{n \geq 0 \mid I \subseteq \mathfrak{m}^n\}.$$

Theorem 3.10

Let $(R, \mathfrak{m})$ be a RLR with $\dim R = 2$, and $|R/\mathfrak{m}| = \infty$. Let Q be a parameter ideal of R s.t. $Q \neq \mathfrak{m}$. Set $I = Q : \mathfrak{m}$. Then TFAE.

(1) *$\mathcal{R}(I)$ is an almost Gorenstein graded ring.*

(2) $\mathrm{o}(Q) \leq 2$.

Thank you so much for your attention.

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